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COMPUTING GENERA OF SATELLITE TUNNEL NUMBER ONE KNOTS AND TORTI-RATIONAL KNOTS

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Abstract

We develop a method to compute the genera and slopes of essential surfaces in 2-bridge link exteriors with one longitudinal boundary component. The tools we use are those developed by Floyd and Hatcher in [4]. Such computations allow us to compute the genera of satellite tunnel number one knots and torti-rational knots.

1. Introduction

A family of knots widely studied is the one known as $(1, 1)$-knots, these are knots which can be put in 1-bridge position with respect to a standard torus in S^3. This family contains all 2-bridge knots, all satellite tunnel number one knots, and it is contained in the family of tunnel number one knots. Genus one and genus two $(1, 1)$-knots have been classified in [9] and [2], respectively. It is natural to ask for a classification of $(1, 1)$-knots of any genus g. Such knots are divided into the satellite and the non-satellite cases. For the non-satellite case we expect to have a description similar to that in [2], as special banding of two $(1, 1)$-knots of smaller genus. In the case that the knot is satellite, we need to determine the 4-tuple α, β, p, q of the Morimoto-Sakuma construction that produces satellite genus g tunnel number one knots [10]. The parameters α, β describe a 2-bridge link $L_{\beta/\alpha}$ and p, q a companion torus knot. These knots are denoted by $K(\alpha, \beta; p, q)$. For genus $g \geq 3$ a minimal genus Seifert surface F may intersect the companion torus in a non-empty collection of longitudes, hence the surface is broken into two pieces, one piece consists of Seifert surfaces for the companion torus, the other piece is a surface $\tilde{F}$ contained in the neighborhood of the torus knot with one boundary parallel to the satellite knot and boundary components which are slopes on the companion torus. Such a surface defines an essential surface F' for the link $L_{\beta/\alpha}$, with one boundary parallel to a component of the link and a number of boundary components on the other component. Floyd and Hatcher [4] classified essential surfaces for 2-bridge links. Later Hoste and Shanahan [7] described an algorithm for computing the slopes of such surfaces. However the calculation of genera of the surfaces is not given there. Furthermore Goda, Hayashi, and Song computed the Euler characteristic of a certain family of such surfaces, see [5].

We were able to determine that an essential surface F' for a 2-bridge link with one boundary on one component of the link and a number of boundary slopes on the other component

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of the link arises from at most two minimal edge-paths of the Floyd-Hatcher construction by means of continued fraction expansions for β/α. This gives a constructive description of the surfaces and allows us to compute genus and slope of the surface as well as to determine whether or not the surface is a fiber of a fibering over the circle for the link.

Applying these ideas to satellite tunnel number one knots we obtain the following result:

Theorem 1. *Let $L_{\beta/\alpha} = K_1 \cup K_2$ be the 2-bridge link given by the tunnel number one satellite knot $K(\alpha,\beta; p,q)$. And let F' be the essential surface for $L_{\beta/\alpha}$ that arises from F, a minimal genus Seifert surface for $K(\alpha,\beta,p,q)$. Suppose $lk(K_1,K_2) \neq 0$. Then*

(1) *If $0 \leq \beta \leq \alpha$, $pq \geq 0$ and $[0; 2n_1, ..., 2n_j]$ is the unique continued fraction for β/α with j odd, the genus of F' is:*

$$\frac{1}{2}\left[\left(-1 + \sum_{k:odd} |n_k|\right)(|lk(K_1,K_2)pq| - 1) + (j+1) - (|lk(K_1,K_2)| + 1)\right]$$

where $k \in \{1, ..., j\}$

(2) *If $0 \leq \beta \leq \alpha$, $pq \leq 0$ and $[1; 2m_1, ..., 2m_i]$ is the unique continued fraction for β/α with i odd, the genus of F' is:*

$$\frac{1}{2}\left[\left(-1 + \sum_{h:odd} |m_h|\right)(|lk(K_1,K_2)pq| - 1) + (i+1) - (|lk(K_1,K_2)| + 1)\right]$$

where $h \in \{1, ..., i\}$.

Corollary 2. *Let $K = K(\alpha,\beta,p,q)$ be a tunnel number one satellite knot such that $lk(K_1,K_2) \neq 0$. Then the genus of K is:*

$$g(K) = g(F') + |lk(K_1,K_2)|\frac{(|p|-1)(|q|-1)}{2}$$

Where F' is as in Theorem 1.

It is worth mentioning that Hirasawa and Murasugi [6] obtained similar results using the Alexander polynomial.

We can also apply our technique to compute the genus of torti-rational knots, which are obtained from a 2-bridge link as follows: Let $L_{\beta/\alpha} = K_1 \cup K_2$ be a 2-bridge link in S^3. Since K_1 is a trivial knot in S^3, K_2 can be considered as a knot in an unknotted solid torus V, the exterior of K_1. A copy of K_1 can be considered as a meridian of V. Then by applying Dehn twists along a meridian disk of V in an arbitrary number of times, say r, we obtain a new knot K from K_2. We call this knot a *torti-rational knot* and it is denoted by $K(\beta/\alpha; r)$.

Theorem 3. *Let $K(\beta/\alpha; r)$ be a torti-rational knot and F a minimal genus Seifert surface for it. Suppose that $lk(K_1,K_2) \neq 0$. Then:*

(1) *Suppose that $r > 1$ and $[1; 2m_1, ..., 2m_i]$ is the unique continued fraction for β/α with i odd, the genus of F is:*

$$\frac{1}{2}\left[\left(-1 + \sum_{h:odd} |m_h|\right)(|lk(K_1,K_2)r| - 1) + (i+1) - (|lk(K_1,K_2)| + 1)\right]$$

where $h \in \{1, ..., i\}$

(2) *Suppose that* $r < -1$ *and* $[0; 2n_1, ..., 2n_j]$ *is the unique continued fraction for* β/α *with j odd, the genus of F is:*

$$\frac{1}{2}\left[\left(-1 + \sum_{k:odd} |n_k|\right)(|lk(K_1, K_2)r| - 1) + (j + 1) - (|lk(K_1, K_2)| + 1)\right]$$

where $k \in \{1, ..., j\}$

(3) *Suppose that* $|r| = 1$ *and* $|lk(K_1, K_2)| > 1$. *Let* $[s; 2r_1, ..., 2r_k]$ *be the continued fraction expansion for* β/α *with* $s = 0$ *or* 1 *such that* $k \geq 3$ *and* $|r_t| \geq 2$ *for all t. The genus of F is:*

$$1 + \frac{(|lk(K_1, K_2)| + 1)(k - 3)}{4}$$

(4) *Suppose that* $|r| = 1$ *and* $|lk(K_1, K_2)| = 1$ *and* $[0; 2n_1, ..., 2n_j]$ *and* $[1; 2m_1, ..., 2m_i]$ *are the continued fraction for* β/α *with* j, i *odd. The genus of F is:*

$$\min\left\{\frac{i-1}{4}, \frac{j-1}{4}\right\}$$

In case that $lk(K_1, K_2) = 0$ we prove:

Theorem 4. *If* $lk(K_1, K_2) = 0$, *the genus of a satellite tunnel number one knot* $K(\alpha, \beta; p, q)$ *is one half the wrapping number of* K_2 *in* $E(K_1)$. *Moreover, if* $[s; 2r_1, ..., 2r_k]$ *is the continued fraction expansion for* β/α *with* $s = 0$ *or* 1 *such that k odd, the genus of* $K(\alpha, \beta, p, q)$ *is* $\Sigma_{i:odd}|r_i|$. *The same is true for a torti-rational knot.*

Theorems 1 and 3 required a description of β/α as a continued fraction and some computations. We have written an algorithm that receives as inputs α, β, p, q, r and outputs genus, slopes and number of boundary components for the surface, in some cases it can be determined the fiberedness of the knot.

Our algorithm is based on that given by Hoste and Shanahan in [7] . We found a fault for rationals $\beta/\alpha > 1/2$, thus it was necessary to reprogram this algorithm to compute the paths and to incorporate computations of genus, slopes and number of boundary components. Our modification of their algorithm can be found at https://github.com/viorato/compute_rational_links_genus.

In Section 2 we review the concepts from the paper [4] of Floyd and Hatcher which are necessary to develop our techniques. In Section 3 we state the basic results that allow to describe the specific type of edge-paths associated to the surfaces of our interest. Using continued fraction expansions for β/α we compute the genera and slopes for the surfaces in Section 3.1. We revisit [4] to give their criteria for a surface to be a fiber of a fibering for a 2-bridge link and give a criteria in terms of the continued fraction expansions for our surfaces to be fibers in Section 4. Finally in Section 5 we compute the genus for satellite tunnel one knots and for torti-rational knots.

2. Preliminaries

2.1. The diagram of slope system in the four puncture sphere. For the sake of helping the reader, in this section we quote literally parts of Section 1 in [4]. Let $L_{\beta/\alpha}$ be a 2-bridge link in S^3, it is represented by a rational number β/α. We may suppose $0 < \beta < \alpha$, α even, and $\gcd(\alpha,\beta) = 1$. Let $n(L_{\beta/\alpha})$ be a regular neighborhood of $L_{\beta/\alpha}$ in S^3. The *exterior of* $L_{\beta/\alpha}$ is $E(L_{\beta/\alpha}) = \overline{S^3 - n(L_{\beta/\alpha})}$. We say that a surface S properly embedded in $E(L_{\beta/\alpha})$ is *essential* if it is incompressible, ∂-incompressible and not boundary parallel. The main idea of Floyd and Hatcher's construction given in [4] is to associate to an essential surface S in $E(L_{\beta/\alpha})$ an edge-path from $1/0$ to β/α in the Diagram D_t, $t \in [0,\infty]$, shown in Figure 1. Observe that $D_0 = D_\infty$.

The diagram D_1 is an embedded graph on the upper half plane $\mathbb{H}$ with the real line $\mathbb{R}$ and the point at infinity $1/0$. Its vertices are the rational points in $\mathbb{R} \cup \{1/0\}$ and its edges are hyperbolic lines in the upper half model of $\mathbb{H}$ joining two vertices a/c, b/d, $(a,b,c,d \in \mathbb{Z})$ if and only if $ad - bc = \pm 1$. These lines are the edges of ideal triangles in $\mathbb{H}$ and $PSL_2(\mathbb{Z})$ is the group of orientation-preserving symmetries of this ideal triangulation. The diagram D_1 is transformed onto the Poincaré disk model by $-\frac{z-\frac{1+i}{2}}{z-\frac{1-i}{2}}$, see Figure 1. Let $G \subset PSL_2(\mathbb{Z})$ be the subgroup of Möbius transformations $(az+b)/(cz+d)$ with c even. Its fundamental domain is the triangle $\langle 1/0, 0/1, 1/1\rangle$. Consider the ideal quadrilateral $Q = \langle 1/0, 0/1, 1/2, 1/1\rangle$. The G-images of this quadrilateral tessellate $\mathbb{H}$. We form the diagram D_0 from D_1 by deleting the G-orbit of the diagonal $\langle 0/1, 1/1\rangle$ of $\langle 1/0, 0/1, 1/2, 1/1\rangle$ and adding the G-orbit of the opposite diagonal $\langle 1/0, 1/2\rangle$. The diagram D_t, $0 < t < \infty$, $t \neq 1$, is obtained from D_1 by deleting the diagonal $\langle 0/1, 1/1\rangle$ in each quadrilateral Q and adding a small rectangle having a vertex in the interior of each edge of Q so that $g(D_t) = D_t$ for $g \in G$. The edges of D_t fall into four G-orbits, labelled A, B, C, D.

REMARK 5. As t approaches to 0 and 1, the inscribed rectangle collapses to the diagonals $\langle 1/0, 1/2\rangle$ and to the diagonal $\langle 0/1, 1/1\rangle$, respectively. See Figure 2.

For a given reduced rational number β/α, let γ denote an oriented edge-path from $1/0$ to β/α in D_t with $0 \leq t \leq \infty$.

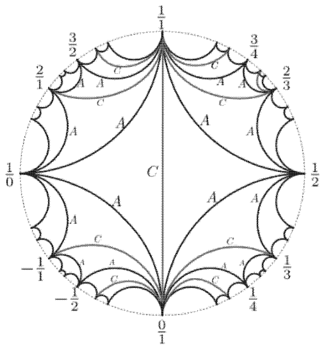
DEFINITION 6. An edge-path γ is called *minimal* if no two consecutive edges in γ lie on the boundary of the same triangle face or rectangle face in D_t.

Then for every minimal edge-path γ in D_t, Floyd-Hatcher construct a corresponding branched surface Σ_γ. Four basic branched surfaces, Σ_A, Σ_B, Σ_C, and Σ_D are assigned to the labelled edges. See Figure 3.

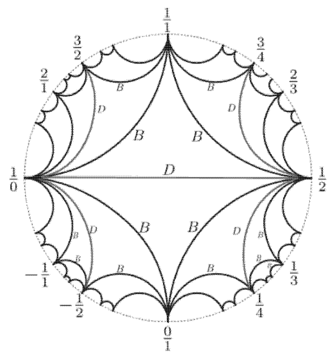
We regard S^3 as the two point compactification of $S^2 \times \mathbb{R}$ and we place the link $L_{\beta/\alpha} \subset S^2 \times I$ so that it meets $S^2 \times \{0\}$ and $S^2 \times \{1\}$ each in two arcs and each intermediate level in four points. We think of each level $S^2 \times \{r\}$ as the quotient $\mathbb{R}^2/\Gamma$, where Γ is the group generated by 180° rotations of $\mathbb{R}^2$ about the integer lattice points $\mathbb{Z}^2$. The four points of the link at each intermediate level are precisely the four points of $\mathbb{Z}^2/\Gamma$. The two arcs at level $r = 1$ have slope β/α and those arcs at level $r = 0$ have slope $1/0$. $PSL_2(\mathbb{Z})$ acts linearly on the level sphere $S^2 \times \{r\} = \mathbb{R}^2/\Gamma$, leaving $\mathbb{Z}^2/\Gamma$ invariant.

The vertices of the diagrams $D_1, D_0 = D_\infty, D_t$ correspond to the slopes of arcs in the level spheres.

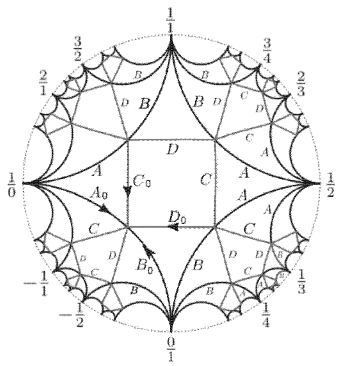
Let $e_1, \ldots, e_k$ be the sequence of edges of a minimal edge-path γ. An edge e_i is the image of one edge $e_0 \in \{A_0, B_0, C_0, D_0\}$ (see Figure 1(c)) under a unique $g_i \in G$. Each e_i determines a branched surface Σ_{e_i} in $S^2 \times [(i-k)/k, i/k]$ and the desired Σ_γ is the union



(a) Diagram D_1



(b) Diagram $D_0 = D_\infty$



(c) Diagram D_t, $t \neq 0, 1, \infty$

Fig. 1.

of these Σ_{e_i}'s. If the orientations of e_i and $g_i(e_0)$ match, then $\Sigma_{e_i} = (g_i \times \varphi)(\Sigma_{e_0})$ where $\varphi : [0,1] \to [(i-1)/k, i/k]$ is $\varphi(t) = \frac{i+t-1}{k}$. If the orientations do not match, we reflect Σ_{e_i} upsidedown. See [5].

Finally, a surface carried by one of the branched surfaces Σ_γ is determined by μ and ρ, the numbers of sheets of the surface along $\partial n(K_1)$ and $\partial n(K_2)$, respectively, and by how the surface branches in each segment Σ_A, Σ_B, Σ_C, or Σ_D of Σ_γ. We set $t = \mu/\rho$, which is the subscript of D_t .

Theorem 3.1(a) of [4] implies that every orientable essential surface in $E(L_{\beta/\alpha})$ with $\mu, \rho \neq$

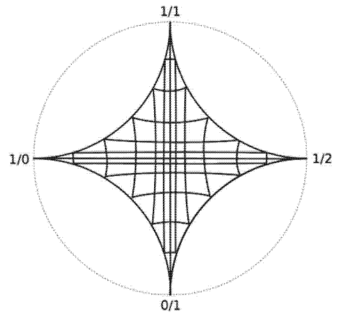


Fig. 2. Collapsing a D_t diagram.

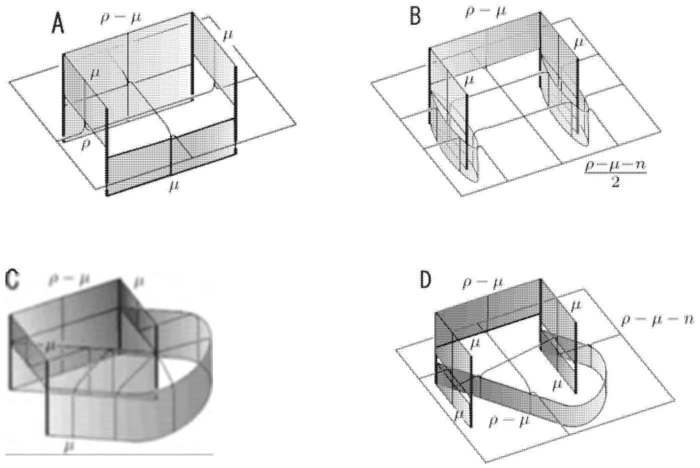


Fig. 3. Branched pieces for edges A, B, C and D

0 is carried by some branched surface corresponding to a minimal edge-path from $1/0$ to β/α in D_t. Conversely, an orientable surface carried by such a branched surface is essential.

A branched surface may carry non-orientable surfaces. Moreover, as noted in [5] there may be an essential non-orientable surface which is not carried by any branched surface.

There is a unique finite sequence of quadrilaterals $Q_{\beta/\alpha}$ such that the first one contains the vertex $1/0$, the last one contains the vertex β/α and every pair of consecutive ones intersects in a single edge.

Remark 7. In a D_t diagram with $t \neq 0, \infty$, the first and the last edges in any edge-path are of type A.

2.2. Edge-paths and essential saddles. Let $S \subset E(L_{\beta/\alpha})$ be a compact orientable essential surface with boundary on $\partial E(L_{\beta/\alpha}) \subset S^2 \times I \subset S^3$.

We may isotope S so that:

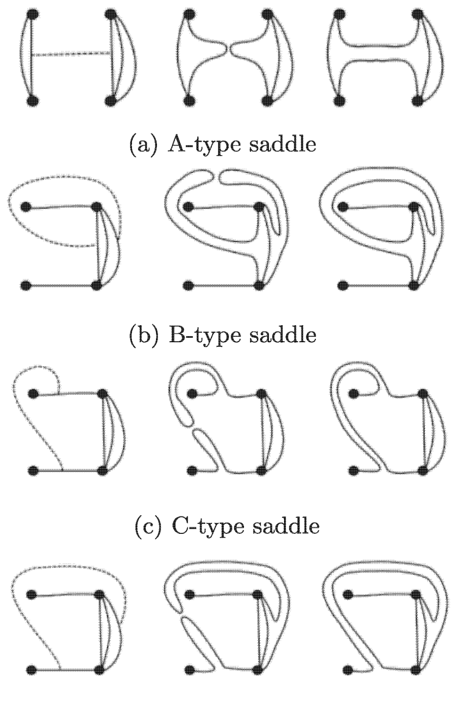
(1) Each component of ∂S is either a meridian of $\partial E(L_{\beta/\alpha})$ in $S^2\times(0,1)$, or is transverse to all meridians of $\partial E(L_{\beta/\alpha})$.
(2) S is transverse to $S^2 \times \partial I$ and lies in $S^2 \times I$ near $E(L_{\beta/\alpha}) \cap (S^2 \times \partial I)$.

(3) The projection $S \cap (S^2 \times I) \to I$ is a Morse function with all its critical points in the interior of S.

Let S^2_r denote $S^2 \times \{r\}$ for $0 < r < 1$. A transverse intersection $S \cap S^2_r$ can contain no arcs which are peripheral in $S^2_r - n(L_{\beta/\alpha})$ in view of (1) and the ∂-incompressibility of S. As r varies from 0 to 1, the point $\lambda_r \in D_t$ can change only at critical levels of the projection $S \cap (S^2 \times I) \to I$, in fact, only at saddles. A saddle where λ_r changes is called an *essential saddle*. So we obtain a finite sequence of λ_r's, say $\lambda_0, ..., \lambda_k$, with $\lambda_{i+1} \neq \lambda_i$ for all i. By (2) λ_0 is the vertex $1/0$ of D_t and λ_k is the vertex β/α.

We can isotope S to lie in $S^2 \times I$ and have all its critical points lying on essential saddles, and also still satisfy (1) – (3) above, see Section 7 of [4].

The possibilities, up to level-preserving isotopy, for an essential saddle corresponding to a segment $\langle \lambda_i, \lambda_{i+1} \rangle$ on an $A-$, $B-$, $C-$ or $D-$type edge of D_t are shown in Figure 4. The two leftmost vertices depict $\partial n(K_2) \cap S^2_r$ and the rightmost vertices depict $\partial n(K_1) \cap S^2_r$.



(a) A-type saddle

(b) B-type saddle

(c) C-type saddle

(d) D-type saddle

Fig. 4. Saddle types: The two leftmost vertices depict $\partial n(K_2) \cap S^2_r$ and the rightmost vertices depict $\partial n(K_1) \cap S^2_r$.

The corresponding saddle to an $A-, B-, C-$ or $D-$type edge of D_t, will be called an $A-, B-, C-$ or $D-$type saddle, respectively.

3. General results

Let $L_{\beta/\alpha} = K_1 \cup K_2$ be a 2-bridge link in S^3 and let $S \subset E(L_{\beta/\alpha})$ be a connected, compact, essential and orientable surface, both as in Section 2.2. Assume that S has n boundary components in $\partial n(K_1)$, which are non-meridional and $n \neq 0$, i.e., μ is a multiple of n, and has one boundary component in $\partial n(K_2)$ parallel to K_2, i.e, $\rho = 1$. Let us denote by $\partial_i S$ the set of boundary components of $\partial n(K_i) \cap S$, for $i = 1, 2$. Observe that $\partial_2 S$ consists only of one curve whose slope is an integer and $\partial_1 S$ of n parallel curves with slope p/q, with respect to a meridian and preferred longitude in each component of the link. We denote the linking number of $L_{\beta/\alpha} = K_1 \cup K_2$ by $lk(K_1, K_2)$.

In the following lemmas we will determine the saddle types corresponding to a minimal edge-path associated to S. Since there is a bijective correspondence between edges, saddles and pieces of branched surfaces, the results can be applied to the three concepts.

Since $\mu/\rho \neq 0, \infty$ and by Remark 7 the first and last saddles are of type A. By Lemma 7.1 and Figure 7.2 of [4], we have the following statement:

Lemma 8. *Suppose that $\mu \neq 1$.*

(1) *$B-$type saddles come in groups of $(\mu - 1)/2$ saddles.*
(2) *$D-$type saddles come in groups of $(\mu - 1)$ saddles.*

Next we will prove that only edges of type A, B and D can occur. Choosing an orientation for S will induce an orientation on the boundary components of S and on the arcs of $S \cap S^2_\epsilon$ for ϵ before the first $A-$type saddle; choose one. When two arcs are being fused by a saddle, in a small neighborhood before the fusion occurs, we see two small arcs with opposite orientations.

Lemma 9. *There are no C-type saddles.*

Proof. At the first level S^2_ϵ, there is only one arc of $S \cap S^2_\epsilon$ connecting the vertices of $\partial n(K_2) \cap S^2_\epsilon$. This implies that in a small neighborhood around one of the vertices of $\partial n(K_2) \cap S^2_\epsilon$, we see only one arc pointing out and around the other vertex we see only one arc pointing in; we see opposite orientations around these vertices. This property must be preserved for all the different levels S^2_r.

If a C-type saddle exists then after a G-transformation, it looks like in Figure 4(c) . But that will imply that the orientations around the vertices $K_2 \cap S^2_r$, at some r, are no longer opposite. □

One crucial object that we used on the proof of Lemma 9 and that we will use is the orientation of $S \cap S^2_\epsilon$ around a small neighborhood of a vertex. Once that we orientate S, it induces an orientation on the arcs $S \cap S^2_\epsilon$ around a vertex, we can assign a $+1$ to each arc pointing out and a -1 to an arc pointing in. We can then compute the sum of the signs around a vertex v, we denote it by Σ_v. Observe that Σ_v is independent of the level S^2_r and it reverses its sign if we change the orientation of S. Thus $|\Sigma_v|$ is a constant that is independent of the level S^2_r and the orientation of S.

Lemma 10. *If the boundary slope of $\partial_1 S$ is of the form p/q with $p, q \in \mathbb{Z} - \{0\}$. Then $|(p/q)\Sigma_v| = |lk(K_1, K_2)|$ for each vertex v in $\partial n(K_1) \cap S^2_\epsilon$*

Proof. Let m_i and l_i be a meridian and preferred longitude of $\partial n(K_i)$, respectevely, for $i = 1, 2$. By definition we can calculate $|\Sigma_v|$ around m_1, computing the intersections with signs of $\partial_1 S$ and m_1. As the slope of $\partial_1 S$ is p/q, each boundary component of $\partial_1 S$ intersects m_1 exactly q times. Let n_+ be the number of the components intersecting positively m and n_- the number of components which intersect m negatively, then $\Sigma_v = q(n_+ - n_-)$.

Now, we only need to prove that $p(n_+ - n_-) = lk(K_1, K_2)$. This can easily be done by observing that S represents an equivalence between $\partial_2 S = km_2 + l_2$ and $\partial_1 S = (n_+ - n_-)(pm_1 + ql_1)$ on $H_1(E(L_{\beta/\alpha}))$. Combining these with the relations $l_1 = lk(K_1, K_2)m_2$ and $l_2 = lk(K_1, K_2)m_1$ we obtain the required equality. □

From the previous proof, it seems that we could get rid of the absolute values from the statement. But the problem is that our definition of Σ_v has an ambiguity on its sign. It is possible to avoid it by being more specific on its definition, but we wouldn't win much it is more convenient to use and compute $|\Sigma_v|$.

Lemma 11. *Suppose that $\mu > 1$, and let S be a surface given by an edge-path in D_t.*

(1) *If there is a B−type saddle, then $|\Sigma_v| = 1$ for all v in $\partial n(K_1) \cap S^2_\epsilon$. Moreover, each boundary component of S on $\partial n(K_1)$ is longitudinal and $\mu = n$.*
(2) *If there is a D−type saddle, then $|\Sigma_v| = \mu$ for all v in $\partial n(K_1) \cap S^2_\epsilon$. Moreover, all the boundary components of S have the same orientation on the boundary $\partial n(K_1)$.*

Proof. (1) By Lemma 8 the number of arcs in $S \cap S^2_\epsilon$ joining the components of $\partial n(K_1) \cap S^2_\epsilon$ is odd. Before a B−type saddle appears, there must be an A−type saddle. After passing it, we see an even number of arcs joining the components of $\partial n(K_1) \cap S^2_\epsilon$. In order to perform a B−type saddle, two arcs of the same slope must be joined, thus their orientation are opposite. Then all the arcs joining the components of $\partial n(K_1) \cap S^2_\epsilon$ can be paired together on opposite orientation pairs. This implies that $|\Sigma_v| = 1$ for each vertex $v \in \partial n(K_1) \cap S^2_\epsilon$.

By Lemma 10 we have that the slope $\partial_1 S = p/q$ is equal to $lk(K_1, K_2)$, hence $\partial_1 S$ is an integer (its components are longitudinal).

(2) After a G transformation, a D−type saddle looks like in Figure 4(d). When performing a D−type saddle, the configuration of arcs that we obtain contains two arcs of slope zero whose orientations coincide with the one on the previous arcs of slope zero. This occurs every time we perform a D−type saddle and by Lemma 8 this happens $\mu - 1$ times, thus the arcs $S \cap S^2_\epsilon$ joining the components of $K_1 \cap S^2_\epsilon$ have the same orientation. Therefore $|\Sigma_v| = \mu$. □

An immediate consequence of Lemmas 11 and 10 is the following.

Corollary 12. *If there is a B−type saddle and if the boundary slope of $S \cap \partial n(K_1)$ equals $1/r$ then $|lk(K_1, K_2)| = 1$ and $r = 1$.*

Summarizing we have:

Corollary 13. *Let $L_{\beta/\alpha} = K_1 \cup K_2$ be a 2-bridge link in S^3 and let $S \subset E(L_{\beta/\alpha})$ be a properly embedded, connected, compact, essential and orientable surface. Assume that S has n boundary components on $\partial n(K_1)$, which are non-meridional and $n \neq 0$, i.e., μ is a multiple of n, and has one boundary component on $\partial n(K_2)$ parallel to K_2, i.e., $\rho = 1$. We have the following:*

(1) *If $\mu > 1$ then the sequence of saddles corresponding to a minimal edge-path associated to S consists only of A− and B−type saddles, or only of A− and D−type saddles.*
(2) *If $\mu = 1$ then the sequence of saddles corresponding to a minimal edge-path associated to S consists only of A−type saddles.*

Proof. If $\mu > 1$, by Lemma 11 we have part (1). In the case that $\mu = 1$, then the corresponding path associate to the surface S lies in the D_1 diagram and there are no $C-$types saddles by Lemma 9, therefore the sequence of saddles are only of type A. □

DEFINITION 14. We will use the notation $AB-$edge-path to refer to an edge-path consisting of only $A-$ and $B-$type saddles. Similarly we use the notation AD- and $A-$edge-path.

By Corollary 13 the only orientable surfaces considered in this article come from $AD-$, $AB-$ or $A-$ edge-paths. Nevertheless not all such edge-paths correspond to an orientable surface.

For instance, consider the edge-path $\langle 1/0, 0/1\rangle$, $\langle 0/1, 1/2\rangle$, $\langle 1/2, 1/3\rangle$, $\langle 1/3, 3/8\rangle$, the corresponding sequence of saddles is $ADAADA$, see Figure 5. In Figure 6 we show the first part of the saddle sequence (recall that we are using $\mu - 1$ type D saddle). Observe that passing to the third saddle of type A gives rise a nonorientable surface.

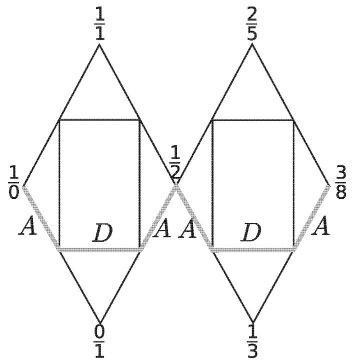


Fig.5. An edge-path from $\frac{1}{0}$ to $\frac{3}{8}$

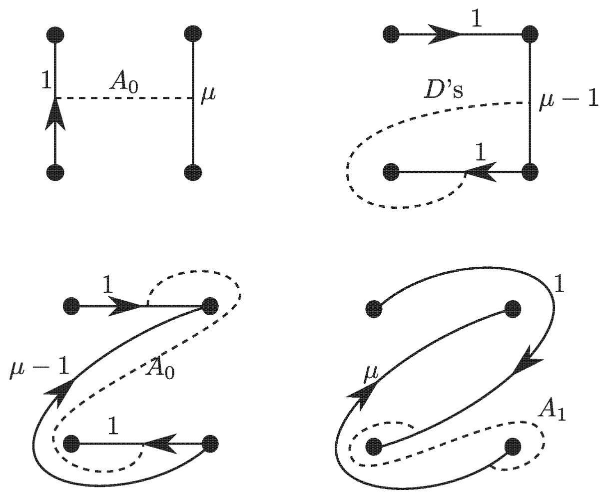


Fig.6. A-type saddles for the edge-path from $\frac{1}{0}$ to $\frac{3}{8}$

The same observation is valid for $AB-$ or $A-$edges-paths, namely there are such edge-paths that correspond to non-orientable surfaces. The next lemma rules out edge-paths corresponding to non-orientable surfaces. In order to state the result we introduce some notation.

Each reduced fraction p/q in $\mathbb{Q}$ can be identified with $0/1$, $1/0$ or $1/1$ by reducing p and q mod 2. An A−type edge in D_t is contained in an edge $\langle p_1/q_1, p_2/q_2\rangle$. If $\{p_1/q_1, p_2/q_2\}$ is identified with $\{0/1, 1/0\}$ mod 2, we say that such an edge is of type A_0. On the other hand if $\{p_1/q_1, p_2/q_2\}$ is identified with $\{1/1, 1/0\}$ mod 2 the edge is said to be of type A_1.

By an A_iX−edge-path we will mean an AX−edge-path in D_t that consists only of edges of type X and A_i with $i = 0, 1$ and $X = B, D$. Similarly we use the notation A_i−edge-path for an edge-path in D_1 that contains only A_i−type edges with $i = 0, 1$.

Lemma 15. *Let S be an orientable surface and γ be an edge-path in D_t associated to S. Suppose that γ is an AX−edge-path with $X = B, D$. Then γ is an A_iX−edge-path with* $i = 0, 1$. *The same result is valid for A−edge-paths.*

Proof. Assume that γ contains edges of type A_0 and A_1. We are going to find a contradiction.

Case 1: γ is an A-edge-path in D_1. As γ is made of only A type saddles, there must be two consecutive saddles of type A_0 and A_1. Without loss of generality, we can assume that A_1 follows A_0. We draw the sequence of pictures mod 2 for these two saddles in Figure 7. Notice that this is impossible due to orientability of S.

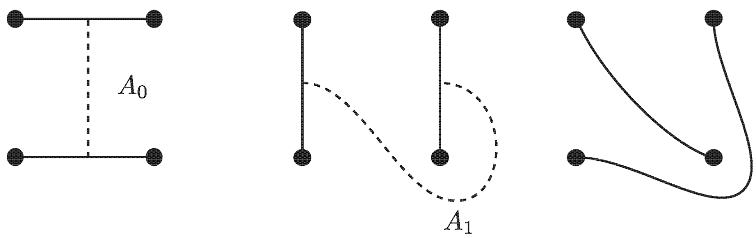


Fig. 7. No orientability of A-edge-path containing both A_0 and A_1 edges.

Case 2: γ is an AD-edge-path in D_t. Again, in this case we will have two consecutive saddles of type A_0 and A_1, because the edge-path comes in blocks of the form $AD\ldots A$ where the two $A's$ are of the same type. The sequence of levels mod 2 is similar to the previous one, see Figure 8, but with some extra $\mu - 1$ parallel arcs.

By Lemma 11(1) those $\mu - 1$ parallel arcs must have all the same orientation; moreover, around the vertices in $\partial n(K_1) \cap S_\epsilon$, all the arcs are oriented in the same direction. It is not hard to see from Figure 8 that it is impossible to give a coherent orientation to all the arcs with the condition that all the μ parallel arcs have the same orientation, contradicting the orientability of S.

Case 3: γ is an AB-edge-path in D_t. A similar phenomenon to the previous case happens here. In fact, we get the same picture as in Figure 8. The reason is that the AB−edge-paths come in blocks of the form $ABBA$ where the two A's are of the same type. So, if we have two A's of a different type on γ, there must be two consecutive blocks with different A-types.

As consequence of Lemma 11(2), all the $\mu - 1$ arcs in the first and last level in Figure 8 need to be cancelled in pairs. And it is impossible to give a coherent orientation satisfying these conditions. □

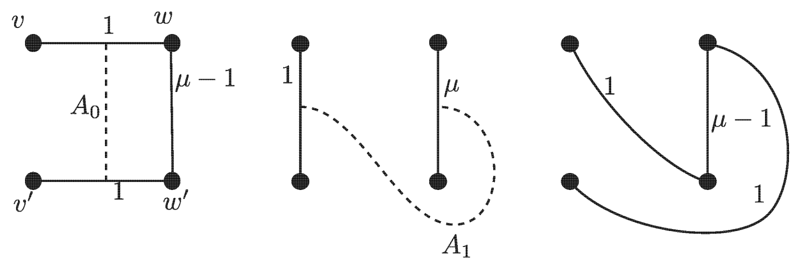
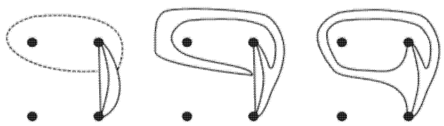


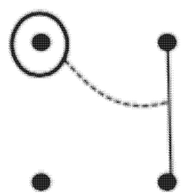
Fig. 8. No orientability of AD-edge-path containing both A_0 and A_1 edges.

REMARK 16. When an $AB-$edge-path happens it must be of the form $ABBA \dots ABBA$, where the $A-$ and $B-$type edges lie in different polygons, see Figure 1(c). Since the surfaces considered in this work are connected, an $AB-$edge-path consists of at least two $ABBA$ blocks.

REMARK 17. In the case that $S \subset E(L_{\beta/\alpha})$ has meridional boundary components on $\partial n(K_1)$ and one boundary component on $\partial n(K_2)$ parallel to K_2, then the edge-path corresponding to the branched surface that carries S belongs to the diagram D_0. Thus it is an $BD-$edge-path. For $B-$type edges to exist and to obtain an orientable surface it must happen that ρ is greater than 1. See Figures 9(a) and 9(b) of $B-$ and $D-$ type saddle for $t = 0$. We conclude that in this case, the edge-path consists only of $D-$type edges.



(a) B-type saddle in D_0



(b) D-type saddle in D_0

Fig. 9.

3.1. Continued fractions and genus of surfaces. An edge-path from $1/0$ to β/α in the diagram D_1 corresponds uniquely to a continued fraction expansion $\beta/\alpha = [r; b_1, ..., b_k]$, where the partial sums $\beta_i/\alpha_i = [r; b_1, ..., b_i]$ are the successive vertices of the edge-path.

$$\frac{\beta}{\alpha} = r + \cfrac{1}{b_1 + \cfrac{1}{b_2 + \cfrac{1}{\ddots + \cfrac{1}{b_k}}}}$$

REMARK 18. At the vertex β_i/α_i the path turns left or right across $|b_i|$ triangles. For i−odd, right if $b_i > 0$ and left if $b_i < 0$. For i−even left if $b_i > 0$ and right if $b_i < 0$. The number of C diagonals is $|b_i|/2$

By Remark 5, in D_t the diagonals C of the diagram D_1 are changed by inscribed rectangles. So for each diagonal C we obtain a D−edge around the vertex β_i/α_i, see Figure 1(a). Thus the number of D−edges around β_i/α_i is $|b_i|/2$.

In this paper we use two special types of continued fraction expansions: $\beta/\alpha = [0; 2n_1, 2n_2, \ldots, 2n_j]$ and $\beta/\alpha = [1; 2m_1, 2m_2, \ldots, 2m_i]$. These are the unique continued fraction where each entry is an even number and j, i are odd.

We will describe the edge-path in D_t associated to these continued fractions, such that the branched surface associated carries a connected, compact, essential and orientable surface S properly embedded in $E(L_{\beta/\alpha})$ with one boundary component on $\partial n(K_2)$ parallel to K_2 and n-boundary components on $\partial n(K_1)$, which are non-meridional and $n \neq 0$; i.e, $\rho = 1$ and μ is a multiple of n. For now on we assume that $t \neq 0, \infty$.

For short we will say that the surface S is associated to the edge-path. We will compute the genus of S as well.

For both continued fraction expansions, the vertices β_i/α_i, given by the partial sums, satisfy that α_{2k+1} is even and α_{2k} is odd.

In the diagram D_1, the edge-path for $[0; 2n_1, 2n_2, \ldots, 2n_j]$ passes by $0/1$ and the edge-path for $[1; 2m_1, 2m_2, \ldots, 2m_i]$ passes by $1/1$. These are A−edge-paths.

The edge-path corresponding to the continued fraction $[0; 2n_1, 2n_2, \ldots, 2n_j]$ is an A_0−edge-path, and the corresponding to the continued fraction $[1; 2m_1, 2m_2, \ldots, 2m_i]$ is an A_1−edge-path.

If $\mu = 1$, the edge-path just obtained is the one that corresponds to S. Hence we obtain an edge-path of length $j+1$ (or $i+1$), where each edge lies in different triangles by construction. For each A−type edge we have an A−type saddle, thus we can compute the genus of S using Euler characteristic.

Proposition 19. *Let $[r; 2r_1, ..., 2r_k]$ be one of the two continued fraction expansions for β/α. If $\mu = 1$, the associated A−edge-path consisting of $k + 1$ edges corresponds to a connected, compact, essential and orientable surface $S \subset E(L_{\beta/\alpha})$ with one boundary component on $\partial n(K_i)$ parallel to K_i for $i = 1, 2$. Then the genus of S is*

$$\frac{1}{2}(k - 1)$$

□

If $\mu \neq 1$, we pass to the D_t diagram with $t \neq 1$. Each edge A in D_1 is changed into an A−edge and a B−edge. The edge path in D_1 is transformed into an AB-edge-path in a diagram D_t. Around a vertex with even denominator there are only A−type edges, and around a vertex with odd denominator there are only $B-$type edges. Thus the pattern $ABBA$ is repeated $\frac{1}{2}(i+1)$-times or $\frac{1}{2}(j+1)$-times.

Observe that an $AB-$edge-path obtained as above may not correspond to a minimal edge-path in D_t, nevertheless a minimal $AB-$edge-path associated to a connected, compact, essential and orientable surface is in correspondence with an A_i-edge-path with $i = 0, 1$. A condition on the continued fraction expansion $[r; 2r_1, ..., 2r_k]$ for β/α for an $AB-$edge-path to be minimal is that $|r_j| > 1$ for all j.

If an orientable surface S is carried by this kind of path, Lemma 11 implies $\mu = n$ and by Remark 16 we have $\frac{1}{2}(i+1) \geq 2$ or $\frac{1}{2}(j+1) \geq 2$, since we require a connected surface, where i, j are the lengths of the continued fraction expansions for β/α. Hence an AB-edge-path that passes trough the vertices $0/1$ or $1/1$ associated to an orientable surfaces must contain at least two blocks of the pattern $ABBA$, thus the continued fraction expansion contains at least three even terms, after the 0 or 1 entries.

In order to compute the genus of S, the associated surface to this edge-path, we count the number of saddles corresponding to the edge-path. Observe that each $A-$type edge corresponds to one saddle and each $B-$type edge to $\frac{1}{2}(n-1)$-saddles. Each block of $ABBA$ contributes with $(n+1)$ saddles. Again, using Euler characteristic we find:

Proposition 20. *Let $[r; 2r_1, ..., 2r_k]$ be one of the two continued fraction expansions for β/α, with $k \geq 3$ and $|r_t| \geq 2$ for all t. If $\mu = n$ and the associated AB-edge-path consisting of $\frac{1}{2}(k-1)$ ABBA blocks corresponds to a connected, compact, essential and orientable surface $S \subset E(L_{\beta/\alpha})$ with one boundary component on $\partial n(K_2)$ parallel to K_2 and n boundary components on $\partial n(K_1)$ parallel to K_1. Then the genus of S is:*

$$1 + \frac{(n+1)(k-3)}{4}$$

□

If S is oriented and $\mu \neq n$ then the edge-path for S is an AD-edge-path. In this case, we substitute each pair BB in the above edge-path by a sequence $DD...D$, where the number of D's is given by the number of diagonals C in the diagram D_1 around the corresponding vertex. For instance, if $\beta_{2k}/\alpha_{2k} = [0; 2n_1, 2n_2, \ldots, 2n_{2k}]$, the number of D's is $|n_{2k+1}|$.

Summarizing, the $AD-$edge-type in D_t associated to the continued fraction expansion $[0; 2n_1, 2n_2, \ldots, 2n_j]$ is $A\underbrace{DD...D}_{|n_1|}AA\underbrace{DD...D}_{|n_3|}AA...AA\underbrace{DD...D}_{|n_j|}A$. Notice that the two consecutive $A-$type edges belong to different triangles, and the $D-$type edges belong to different quadrilaterals by construction. Thus we obtain a minimal edge-path. Analogously, for the continued fraction expansion $[1; 2m_1, 2m_2, \ldots, 2m_i]$ we associate an $AD-$edge-path.

Next we compute the genus of such S.

Proposition 21. *Let $[r; 2r_1, ..., 2r_k]$ be one of the two continued fraction expansions for β/α. If $\mu \neq n$ and the associated path $A\underbrace{DD...D}_{|r_1|}AA\underbrace{DD...D}_{|r_3|}AA...AA\underbrace{DD...D}_{|r_k|}A$ corresponds to a connected, compact, essential and orientable surface $S \subset E(L_{\beta/\alpha})$ with one boundary component on $\partial n(K_2)$ parallel to K_2 and n non-meridional boundary components on $\partial n(K_1)$. Then the genus of S is:*

$$\frac{1}{2}\left[\left(-1+\sum_{h:odd}|r_h|\right)(|\mu|-1)+(k+1)-(n+1)\right]$$

where $h \in \{1, ..., k\}$

Proof. Use Euler characteristic, considering that each A−type edge corresponds to one saddle and each D−type edge corresponds to $\mu - 1$ saddles. □

3.2. Boundary slopes. The boundary of a branched surface derived from the Floyd-Hatcher construction defines a train track on the boundary of the regular neighborhood of the link. Thus the boundary of any essential surface S carried by the branched surface is carried by this train track. Lash, [8], calculated the space of boundary slopes for the Whitehead link.

In the following paragraph we explain Lash algorithm. We base the explanation on the article [5]:

To compute the boundary slopes of the surfaces the frame used consists of the meridian μ_i and a non-standard longitude λ_i of $\partial n(K_i)$. In $S^2 = \mathbb{R}^2/\Gamma$, we take the arc s of slope 0 connecting $\Gamma(0,0)$ and $\Gamma(0,1)$. λ_1 is the union of the arc $(s \times [0,1]) \cap \partial n(K_1)$ and an arc in $(S^2 \times [1,\infty)) \cap \partial n(K_1)$. λ_1 is oriented toward increasing $r \in [0,1]$ along the axis $\Gamma(0,0)\times[0,1]$. The meridian μ_1 is oriented as a right-handed circle around the axis $\Gamma(0,0) \times [0,1]$ oriented upward. We obtain λ_2, μ_2 from λ_1, μ_1 by rotating by 180° about the axis $\Gamma(1/2, 1/2) \times [0,1]$.

Let i_j be the algebraic intersection number $\partial S \cdot \lambda_j$ in $\partial n(K_j)$. Let φ be the map such that for $s \in [0,1]$, $\varphi(s) = (i+s-1)/k \in [(i-1)/k, i/k]$. Recall that in Section 2 a surface $S \subset E(L_{\beta/\alpha})$ corresponds to a minimal edge-path γ with edges $e_1, ..., e_k$ each one is the image of an edge $e_0 \in \{A_0, B_0, C_0, D_0\}$ under a unique $g \in G$, and it is associated to a branched surface Σ_{e_0}. For $0 \le t < 1$, $\partial\Sigma_{e_i} = \partial(g \times \varphi)(\Sigma_{e_0})$, $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$ contributes to the number i_j as in Table 1, if the orientations of e_i and $g(e_0)$ agree. If they disagree, we change the signs of the number in Table 1.

We calculate the boundary slope of a surface $S \subset E(L_{\beta/\alpha})$ corresponding to an AD-edge-path.

Taking the sum of the entries of the row of i_1 and i_2 of Table 1, we can see that the slope on $\partial n(K_1)$ is $(\mu, r(\mu - \rho) + s\rho)$ and that on $\partial n(K_2)$ is $(\rho, v(\mu - \rho) + s\rho)$, where the first coordinate is the longitudinal entry, the second coordinate is the meridional entry with respect to the unusual longitude λ_i. The parameters r, s, v are integer numbers and r is the total contribution of $(\mu - \rho)$ given by D-edges in column i_1, v is the total contribution of $(\mu - \rho)$ given by D-edges in column i_2 and s is the total contribution of ρ given by A-edges in each column. To obtain the real slope, we need to know the slope of the preferred longitude, which is obtained by substituting 1 for μ and 0 for ρ in $\partial n(K_1)$ and turns out to be $(1, r)$. The preferred longitude of K_2 is of slope $(1, s - v)$, which is obtained by substituting $\rho = 1$ and $\mu = 0$ in $\partial n(K_2)$. But the preferred longitude of K_2 is the same for K_1, recall that we take λ_2 as the image of λ_1 by rotating 180° about the axis $\Gamma(1/2, 1/2) \times [0,1]$. Thus, $(1, r) = (1, s - v)$ and $s - r = v$. The slopes with respect to the preferred longitude can be obtained from $(\mu, r(\mu-\rho)+s\rho-r\mu) = (\mu, (s-r)\rho) = (\mu, v\rho)$ on $\partial n(K_1)$, and $(\rho, v(\mu-\rho)+s\rho-(s-v)\rho) = (\rho, v\mu)$ on $\partial n(K_2)$.

Recall that we are considering two types of continued fraction expansions for β/α, namely

Table 1.

Label	condition on $-d/c$	i_1	i_2
A	$-\infty < -\frac{d}{c} < 0$	ρ	ρ
	$0 < -\frac{d}{c} < \infty$	$-\rho$	$-\rho$
	$-\frac{d}{c} = 0, \pm\infty$	0	0
B	$-\infty < -\frac{d}{c} < 0$	$-(\mu-\rho)$	0
	$0 < -\frac{d}{c} < \infty$	$\mu-\rho$	0
	$-\frac{d}{c} = 0, \pm\infty$	0	0
C	$0 < -\frac{d}{c} < 1$	-2ρ	0
	$-\frac{d}{c} = 0, 1$	$-\rho$	ρ
	otherwise	0	2ρ
D	$\frac{1}{2} < -\frac{d}{c} < \infty$	$\mu-\rho$	$\mu-\rho$
	$-\frac{d}{c} = \frac{1}{2}, \pm\infty$	0	$\mu-\rho$
	otherwise	$-(\mu-\rho)$	$\mu-\rho$

$F_0 = [0; 2n_1, 2n_2, \ldots, 2n_j]$ and $F_1 = [1; 2m_1, 2m_2, \ldots, 2m_i]$. As discussed in Section 3.1, for each continued fraction there is an AD-edge-path corresponding to an essential surface. We will determine the contribution of v.

The edge path for F_0 is $A\underbrace{DD...D}_{|n_1|}AA\underbrace{DD...D}_{|n_3|}AA...AA\underbrace{DD...D}_{|n_j|}A$, the orientations of the edges A and D need to be determined in order to compute v. If the orientation of $e_i \in \{A, D\}$ and $g(e_0)$ with $e_0 \in \{A_0, D_0\}$ agree we denote the edge by $\overrightarrow{e_i}$, if they disagree we denote it by $\overleftarrow{e_i}$.

By the construction of the edge path it is not hard to see, as shown in Figure 10, that:

(1) The first A−type edge is of type $\overrightarrow{A}$.
(2) The first $|n_1|$ D−type edges are of type $\overleftarrow{D}$.
(3) Each intermediate pair AA is of the form $\overleftarrow{A}\,\overrightarrow{A}$.
(4) The last A−type edge is of type $\overleftarrow{A}$

For the remaining D−type edges we have:

Proposition 22. *For the continued fraction expansion F_0 and i odd.*

(1) *If $n_i > 0$ then the sequence of D−type edges are of type $\overleftarrow{D}$.*
(2) *If $n_i < 0$ then the sequence of D−type edges are of type $\overrightarrow{D}$.*

Proof. For both cases we need to verify the agreement or disagreement of the D−types edges with $g(D_0)$ at the i−th position for i odd. Since we are considering the continued fraction F_0 all the vertices β_i/α_i, for i odd, are congruent with $0/1$ mod 2, up to transformations of elements of $PSL(2, \mathbb{Z})$. Thus the D−type edge at such vertex β_i/α_i is of type $\overleftarrow{D}$ as shown in Figure 10. From Remark 18 the quadrilateral turns right if $n_i > 0$ and left if $n_i < 0$. Hence, if $n_i > 0$ the sequence of D−type edges are are of type $\overleftarrow{D}$ and if $n_i < 0$ the sequence of D−type edges of type $\overrightarrow{D}$. See Figures 10 and 11 for the turns around β_i/α_i mod 2. □

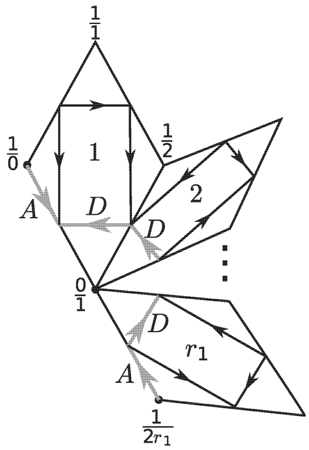


Fig. 10. Path $\overrightarrow{A}\overleftarrow{D}\overleftarrow{D}\dots\overleftarrow{D}\overleftarrow{A}$

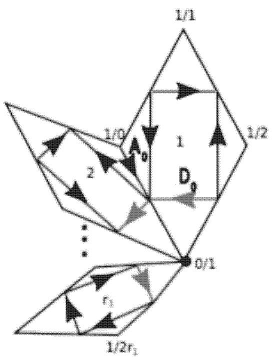


Fig. 11. Path $\overrightarrow{D}\overrightarrow{D}\cdots\overrightarrow{D}$

The value of v for the edge path corresponding to the continued fraction expansion F_0 is $v = -(n_1 + n_3 + ... + n_j)$ because when $n_i > 0$ we see $\overleftarrow{D}$−type edges, so the contribution in the Table 1 is $-n_i$, and if $n_i < 0$ we see $\overrightarrow{D}$−type edges, so they contribute with $-n_i$ in Table 1. Since $lk(K_1, K_2) = n_1 + n_3 + ... + n_j$, we conclude the following:

Corollary 23. *Let $S \subset E(L_{\beta/\alpha})$ be a surface associated to the edge path $A\underbrace{DD...D}_{|n_1|}AA\underbrace{DD...D}_{|n_3|}AA...AA\underbrace{DD...D}_{|n_j|}A$, arising from $[0; 2n_1, 2n_2, \dots, 2n_j]$. The boundary slopes of S with respect to the preferred longitude on $\partial n(K_1)$ is $(\mu, -lk(K_1, K_2)\rho)$ and on $\partial n(K_2)$ is $(\rho, -lk(K_1, K_2)\mu)$.*

On the other hand, for the continued fraction expansion $F_1 = [1; 2m_1, 2m_2, \dots, 2m_i]$ the corresponding edge path in the diagram D_t is $A\underbrace{DD...D}_{|m_1|}AA\underbrace{DD...D}_{|m_3|}AA...AA\underbrace{DD...D}_{|m_i|}A$. This path lies in the same sequence of quadrilaterals as the corresponding path for the continued fraction expansion F_0, but it is made of the $A-$ and $D-$type edges which do not belong to the path for F_0. Reasoning as before, we have that for the $AD-$edge-path corresponding to F_1:

(1) The first $A-$type edge is of type $\overrightarrow{A}$.
(2) The first $|n_1|$ $D-$type edges are of type $\overrightarrow{D}$.
(3) Each intermediate pair AA is of the form $\overleftarrow{A}\,\overrightarrow{A}$.
(4) The last $A-$type edge is of type $\overleftarrow{A}$

Proposition 24. *For the continued fraction expansion F_1 and i odd.*

(1) *If $n_i > 0$ then the sequence of $D-$type edges are of type $\overrightarrow{D}$.*
(2) *If $n_i < 0$ then the sequence of $D-$type edges are of type $\overleftarrow{D}$.*

Corollary 25. *Let $S \subset E(L_{\beta/\alpha})$ be a surface associated to the edge path $A\underbrace{DD...D}_{|m_1|}AA\underbrace{DD...D}_{|m_3|}AA...AA\underbrace{DD...D}_{|m_i|}A$, arising from $[1; 2m_1, 2m_2, \ldots, 2m_i]$. The boundary slopes of S with respect to the preferred longitude on $\partial N(K_1)$ is $(\mu, lk(K_1, K_2)\rho)$ and on $\partial N(K_2)$ is $(\rho, lk(K_1, K_2)\mu)$.*

Analogously, we can compute the boundary slopes for A-edge-path and AB-edge-path, in both cases the resulting boundary slopes are equal to zero.

4. Fiberings

Floyd and Hatcher give a criterion to determine when a surface S in $E(L_{\beta/\alpha})$ is a fiber of a fibering $E(L_{\beta/\alpha}) \to S^1$.

DEFINITION 26. Let γ be a path in D_t, with $t \in [0, \infty]$. A maximal sequence of consecutive $A-$ and $D-$type edges in γ each separated from the next by only one edge in D_t, is called a *string*.

Figure 12(a) shows an example of a string and Figure 12(b) depicts a path which is not a string.

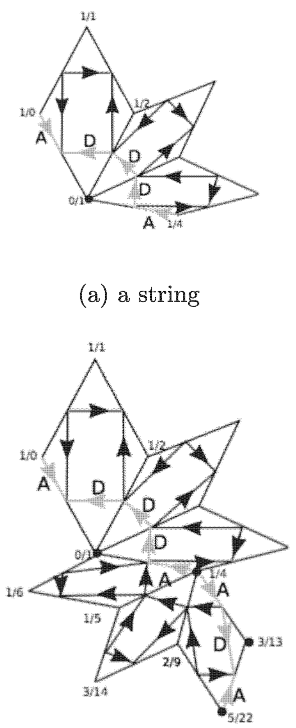
Proposition 6.1 of [4] states sufficient and necessary conditions for fibering:

Proposition 27. *A surface $S \subset E(L_{\beta/\alpha})$ is a fiber of a fibering $E(L_{\beta/\alpha}) \to S^1$ if and only if it is isotopic to a surface carried by a branched surface Σ_γ whose associated edge-path γ from $1/0$ to β/α, in a determined D_t, consists of a single string of $A-$ and $D-$ type edges.*

The following theorem tells us conditions on the continued fraction expansion, considered in this work, for a surface S to correspond to a fiber of a fibering $E(L_{\beta/\alpha}) \to S^1$.

Theorem 28. *Let $L_{\beta/\alpha}$ be a 2-bridge link and S a connected, compact, essential and orientable surface in $E(L_{\beta/\alpha})$ with one boundary component on $\partial n(K_2)$ parallel to K_2 and n-boundary components on $\partial n(K_1)$, which are non-meridional and $n \neq 0$.*

(1) *Suppose S is associated to an AD-edge-path. S is a fiber of a fibering $E(L_{\beta/\alpha}) \to S^1$ if and only if the continued fraction expansion for β/α has the form $[r; 2r_1, 2\epsilon_2, 2r_3, ..., 2\epsilon_{n-1}, 2r_n]$ with $r = 0, 1$ and $|\epsilon_i| = 1$.*
(2) *Suppose S is associated to an A-edge-path. S is a fiber of a fibering $E(L_{\beta/\alpha}) \to S^1$ if and only if the continued fraction expansion for β/α has the form $[r; 2\epsilon_1, 2\epsilon_2, ..., 2\epsilon_n]$ with $r = 0, 1$ and $|\epsilon_i| = 1$ for all i.*
(3) *Suppose S is associated to a D-edge-path. S is a fiber of a fibering $E(L_{\beta/\alpha}) \to S^1$ if*



(a) a string

(b) not a string

Fig. 12.

and only if the continued fraction expansion for β/α *has the form* $[0; 2r_1, -2, 2r_2, ...-2, 2r_n]$ *with* $2r_i$ *positive for all i. Thus the fraction starting with 1 is of the form* $[1; 2n_1, 2, 2n_2, ..., 2, 2n_j]$ *with* $2n_k$ *negative for all k.*

Proof. In each case we need to verify that the corresponding path in the adequate diagram D_t is a string.

(1) Let $\gamma = A\underbrace{DD...D}_{|r_1|}AA\underbrace{DD...D}_{|r_3|}AA...AA\underbrace{DD...D}_{|r_k|}A$ be the edge-path arising from the continued fraction expansions $[r; 2r_1, 2r_2, \dots, 2r_k]$. Observe that any two consecutive A and D are separated by exactly one $C-$type edge in D_t, with $t \neq 0, 1, \infty$. And any two consecutive $D-$type edges are separated by exactly one $A-$ or $B-$type edge as in Figures 12(a) and 12(b). To guarantee that γ is a single string, it is necessary to check when two consecutive $A-$type edges are separated by only one edge. By inspecting Figure 12(b), it is easy to observe that two $A-$type edges are separated by only one edge if $2r_2 = 2, -2$. This pattern is extended to the whole path γ. Thus, the condition is that $r_l = 2\epsilon$ with $\epsilon = -1, 1$ for l even in the continued fraction expansion $[r; 2r_1, 2r_2, \dots, 2r_k]$.

(2) Consider the continued fraction expansion $[r; 2r_1, ..., 2r_j]$, since S is associated an A-edge-path γ in the D_1 diagram, γ goes through all the vertices $1/0, \beta_0/\alpha_0, \beta_1/\alpha_1, ...,$ $\beta_j/\alpha_j = \beta/\alpha$. For γ to be a string, every two consecutive A-type edges must be separated by exactly one C-type edge or by exactly one A-type edge. There are two

possibilities depicted in Figures 13(a) and 13(b), we see that $|2r_i| = 2$ for all i. Thus, the continued fraction expansion has the form $[r; 2\epsilon_1, 2\epsilon_2, ..., 2\epsilon_n]$ with $\epsilon_i = \pm 1$.

(3) Let us consider the continued fraction expansion $[0; 2r_1, ..., 2r_j]$, in this case the surface S is in correspondence with a D-edge-path γ in the D_0 diagram. The first r_1 edges of type D pass through vertices $1/0, 1/2, 1/4, ...1/2r_1 = \beta_1/\alpha_1$. Each two consecutive D-type edges are separated by exactly one B-type edge. Thus, that piece of γ satisfies the condition to be a string. See Figure 14(a). A similar phenomenon occurs around a vertex β_i/α_i with i even. It is necessary to determine when two consecutive D-edges with common vertex β_i/α_i with i odd are separated by exactly one B-edge.

Next we will determine conditions for r_2, r_3 in order to keep γ being a string, up to $PSL_2(\mathbb{Z})$ transformation, we will be able to argue that the conditions for r_2, r_3 can be extended to the following $r_i's$.

First let us consider $2r_2, 2r_3$ both positive. The B-edge connecting $0/1$ and $1/2r_1$ has to turn left $2r_2$ edges to reach the vertex β_2/α_2. Then the edge connecting $1/2r_1$ and β_2/α_2 has to turn right $2r_3$ edges to reach vertex β_3/α_3. Recall that the turns at each vertex were described in Remark 18. For this case see Figure 14(a). The two consecutive D-edges with common vertex $1/2r_1$ are separated by $(2r_2 + 2r_3 - 1)$ B-edges, since $2r_2, 2r_3 \geq 2$, there are at least three B-edges in between. Hence, this situation will not give a string.

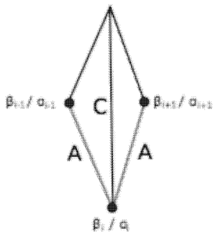
Secondly consider $2r_2$ positive and $2r_3$ negative. In this case, the B-edge connecting $0/1$ and $1/2r_1$ has to turn left $2r_2$ edges to reach the vertex β_2/α_2. Then the edge connecting $1/2r_1$ and β_2/α_2 has to turn left $2r_3$ edges to reach vertex β_3/α_3. The two consecutive D-edges with common vertex $1/2r_1$ are separated by $2r_2$ B-edges, since $2r_2 \geq 2$, there are at least two B-edges in between. Hence, this situation will not give a string. See Figure 14(b).

Thirdly suppose $2r_2$ and $2r_3$ are negative. The B-edge connecting $0/1$ and $1/2r_1$ has to turn right $2r_2$ edges to reach the vertex β_2/α_2. Then the edge connecting $1/2r_1$ and β_2/α_2 has to turn left $2r_3$ edges to reach vertex β_3/α_3. See Figure 14(c). The two consecutive D-edges with common vertex $1/2r_1$ are separated by $(|2r_2| + |2r_3| - 1)$ B-edges, since $2r_2, 2r_3 \geq -2$, there are at least three B-edges in between. Thus, this case will not give a string.

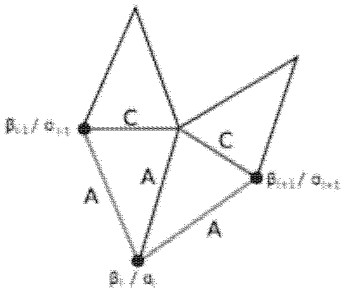
Finally, if $2r_2$ is negative and $2r_3$ is positive. The B-edge connecting $0/1$ and $1/2r_1$ has to turn right $2r_2$ edges to reach the vertex β_2/α_2. Then the edge connecting $1/2r_1$ and β_2/α_2 has to turn right $2r_3$ edges to reach vertex β_3/α_3. See Figure 14(d). In this case the edges with common vertex $1/2r_1$ are separated by $(|2r_2| - 1)$ B-edges, so to obtain a string it is necessary that $2r_2 = -2$.

At this point we have that the continued fraction expansion looks like $[0; 2r_1, -2, 2r_3, x_4, ..., x_n]$.

Using a transformation in $PSL_2(\mathbb{Z})$, we can put in correspondence $\beta_1/\alpha_1 \to 1/0$, $\beta_2/\alpha_2 \to 0/1$ and $\beta_3/\alpha_3 \to \beta_1/\alpha_1$. Analysing as above we are able to conclude that $2r_4 = -2$ and $2r_5$ is positive. Thus, if we keep doing the correspondence for the remaining vertices, we conclude that the continued fraction expansion has the form $[0; 2r_1, -2, 2r_3, -2, ..., -2, 2r_n]$ with $2r_i$ positive for all i odd. A similar analysis shows that the other continued fraction expansion must be $[1; 2n_1, 2, 2n_2, ..., 2, 2n_j]$ with $2n_k$ negative for all k. □



(a) $2r_2$ and $2r_3$ positive



(b) $2r_2$ positive and $2r_3$ negative

Fig. 13. Possibilities for A-edges in D_1 to belong to a string.

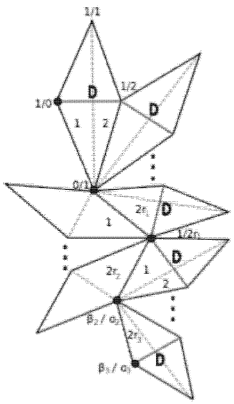
Corollary 29. *Let $L_{\beta/\alpha} = K_1 \cup K_2$ be a 2-bridge link with $lk(K_1, K_2) = 0$. A surface $S \subset E(L_{\beta/\alpha})$ associated to a D-edge-path is not a fiber of a fibering $E(L_{\beta/\alpha}) \to S^1$.*

Proof. The third part of Theorem 28 implies that if the the surface S is carried by a D-edge-path, then the continued fraction expansion for β/α is of the form $[0; 2r_1, -2, 2r_2, ... -2, 2r_n]$ with $2r_i$ positive for all i. Thus the linking number is not equal to zero, a contradiction. □

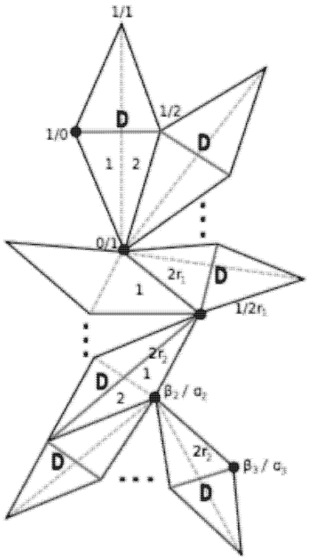
5. Applications

In this section we compute the genus of tunnel number one satellite knots, as well as torti-rational knots. Hirasawa and Murasugi, [6] have computed the genus of such knots using algebraic techniques, namely the Alexander polynomial. We give criteria to determine fiberedness of satellite tunnel number one knots only when $lk(K_1, K_2) \neq 0$.

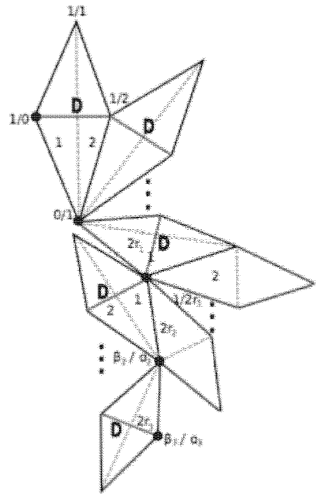
5.1. Tunnel number one satellites knots. Morimoto and Sakuma [10] determined the knot types of satellite tunnel number one knots in S^3. These knots are constructed as follows. Let K_0 be a (p, q)-torus knot in S^3 with $p \neq 1$ and $q \neq 1$, and let $L_{\beta/\alpha} = K_1 \cup K_2$ be a 2-bridge link in S^3 with $\alpha \geq 4$. Note that K_0 is a non-trivial knot, and $L_{\beta/\alpha}$ is neither a trivial link nor a Hopf link. Since K_1 is the trivial knot in S^3, there is a an orientation preserving homeomorphism $f : E(K_1) \to N(K_0)$ which takes a meridian $m_1 \subset \partial E(K_1)$ of K_1



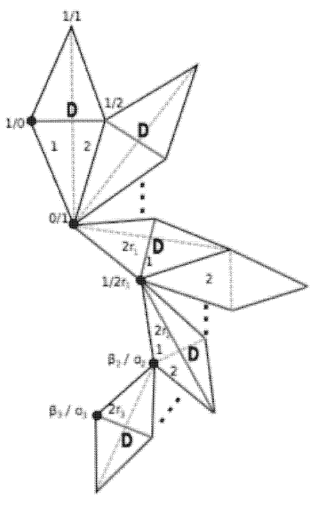
(a) $2r_2$ and $2r_3$ positive



(b) $2r_2$ positive and $2r_3$ negative



(c) $2r_2$ and $2r_3$ negative



(d) $2r_2$ negative and $2r_3$ positive

Fig. 14. Analyse of D-edge-path in D_0 to belong to a string.

to a fiber $h \subset \partial N(K_0) = \partial E(K_0)$ of the unique Seifert fibration of $E(K_0)$. The knot $f(K_2) \subset N(K_0) \subset S^3$ is denoted by the symbol $K(\alpha,\beta;p,q)$. Every satellite knot of tunnel number one has the form $K(\alpha,\beta;p,q)$ for some integers α,β,p,q. Eudave-Muñoz [3] obtained another description of these knots.

Let l and m be a preferred longitude and a meridian for $\partial N(K_0)$, respectively. Notice that $\Delta(l,h) = pq$ and then $\Delta(f^{-1}(l),m_2) = pq$, where Δ stands for the geometric intersection of two curves.

The next lemma is a variation of Lemma 2.11 in [1], applied to our context.

Lemma 30. *Let $K = K(\alpha,\beta;p,q)$ be a satellite tunnel number one knot. Let F be a minimal genus Seifert surface for K. The surface F can be isotoped in such a way that $F \cap \partial N(K_0)$ consists of $|lk(K_1,K_2)|$ preferred longitudes and $F \cap (S^3 - N(K_0))$ is made of $|lk(K_1,K_2)|$ components which are Seifert surfaces for K_0.*

First we consider the case when $lk(K_1,K_2) = 0$.

Suppose the 2-bridge presentation of $L_{\beta/\alpha}$ is given relative to some 2-sphere S in S^3 bounding 3-balls W_0, W_1 such that $L_{\beta/\alpha}$ intersects S transversely and $L_{\beta/\alpha} \cap W_i$ is a disjoint union of two arcs. Consider $S \times I$ be a product regular neighborhood of S in S^3, and let $h : S \times I \to I$ be the height function. We denote the level surfaces $h^{-1}(r) = S \times \{r\}$ by S_r for each $0 \le r \le 1$. S_0 bounds a 3-ball H_0, and S_1 bounds a 3-ball H_1, such that $S^3 = H_0 \cup (S \times I) \cup H_1$. Assume that $S_0 \subset W_0$, $S_1 \subset W_1$, and that $h|(S \times I) \cap L_{\beta/\alpha}$ has no critical points (so $(S \times I) \cap L_{\beta/\alpha}$ consists of monotone arcs).

Let F be an essential surface properly embedded in the exterior $E(L_{\beta/\alpha})$.

By general position, an essential surface can always be isotoped in $E(L_{\beta/\alpha})$ so that:

- **(M1):** F intersects $S_0 \cup S_1$ transversely. We denote the surfaces $F \cap H_0$, $F \cap H_1$, $F \cap (S \times I)$ by F_0, F_1, $\tilde{F}$, respectively;
- **(M2):** each component of ∂F is either a level meridian circle of $\partial E(L_{\beta/\alpha})$ lying in some level set S_r or it is transverse to all the level meridians circles of $\partial E(L_{\beta/\alpha})$ in $S \times I$;
- **(M3):** for $i = 0, 1$, any component of F_i containing parts of $L_{\beta/\alpha}$ is a cancelling disk for some arc of $L_{\beta/\alpha} \cap H_i$. In particular such cancelling disks are disjoint from any arc of $L_{\beta/\alpha} \cap H_i$ other than the one they cancel;
- **(M4):** $h|\tilde{F}$ is a Morse function with a finite set $Y(F)$ of critical points in the interior of $\tilde{F}$, located at different levels. In particular $\tilde{F}$ intersects each noncritical level surface transversely.

We define the complexity of any surface satisfying (M1)–(M4) as the number

$$c(F) = |\partial F_0| + |\partial F_1| + |Y(F)|,$$

where $|Z|$ stands for the number of elements in the finite set Z, or the number of components of the topological space Z.

We say that F is *meridionally incompressible* if whenever F compresses in S^3 via a disk D with $\partial D = D \cap F$ and D intersects $L_{\beta/\alpha}$ in one point interior to D, then ∂D is parallel in F to some boundary component of F which is a meridian circle in $\partial E(L_{\beta/\alpha})$. Otherwise F is *meridionally compressible*. Observe that if F is essential and meridionally compressible then a meridional surgery on F produces a new essential surface in $E(L_{\beta/\alpha})$.

The following is Lemma 3.2 of [11].

Lemma 31. *Let F be a surface in S^3 spanned by K_2 (orientable or not) and transverse to K_1, such that $F' = F \cap E(L_{\beta/\alpha})$ is essential and meridionally incompressible in $E(L_{\beta/\alpha})$. If F' is isotoped so as to satisfy* (M1)–(M4) *with minimal complexity $c(F)$, then $|Y(F')| = 2 - (\chi(F') + |\partial F'|)$ and*

(1) *each critical point of $h|\tilde{F}$ is a saddle,*
(2) *for $0 \leq r \leq 1$ any circle of $S_r \cap F$ is nontrivial in $S_r - E(L_{\beta/\alpha})$ and F, and*
(3) *F_0 and F_1 each consists of one cancelling disk.*

When $lk(K_1, K_2) = 0$, Lemma 30 implies that $F' = f^{-1}(F) \subset E(K_1)$. Moreover F' is an incompressible genus g Seifert surface for K_2.

Lemma 32. *The surface F' can be meridionally compressed g-times to obtain a disk Σ that satisfies the conditions of* Lemma 31. *And g is one half the wrapping number of K_2 with respect to $E(K_1)$. Moreover, if $[s; 2r_1, ..., 2r_k]$ is the continued fraction expansion for β/α with $s = 0$ or 1 such that k odd, the genus of $K(\alpha,\beta, p, q)$ is $\Sigma|r_i|$.*

Proof. We will proceed by induction on the pair $(g(F'), |Y(F')|)$. By Lemma 21 of [2] we know that a surface S with $(g(S), |Y(S)|) \leq (2, 4)$ meridionally compresses $g(S)$-times to a disk satifiying Lemma 31. Let us assume that the result is true for any surface S with $(g(S), |Y(S)|) \leq (g(F'), |Y(F')|)$. Suppose that F' is meridionally incompressible, we can apply Lemma 31, and using the same arguements in Lemma 21 of [2], we obtain a contradiction and thus F' must be meridionally compressible. Moreover after performing the meridional compression a connected surface F^2 is obtained, and $g(F^2) = g(F') - 1$ and $|Y(F^2)| = |Y(F')| - 2$. By induction hypothesis F^2 compresses meridionally $g(F^2)$ times to a disk satisfying Lemma 31. But F^2 was obtained by compressing F' once, thus F' compresses meridionally $g(F')$ times to the required disk Σ. Thus K_2 spans Σ which intersects meridionally K_1 in $2g(F')$ points, this implies that the wrapping number of K_2 in the solid torus $E(K_1)$ is equal to $2g(F')$. Now, to recover F' from Σ we must attached $g(F')$ tubes, therefore the last part of the statement is true. □

Theorem 33. *If $lk(K_1, K_2) = 0$, the genus of a satellite tunnel number one knot is one half the wrapping number of K_2 in $E(K_1)$.*

Proof. Since $lk(K_1, K_2) = 0$ a minimal genus Seifert surface F for $K(\alpha,\beta; p, q)$ of genus g determines a minimal genus Seifert surface $F' = f^{-1}(F)$ of genus g for K_2 in $E(K_1)$. Lemma 32 implies that genus of F' is one half the wrapping number of of K_2 with respect to $E(K_1)$. And the genus of F equals the genus of F'. □

Next we consider the case when $lk(K_1, K_2) \neq 0$.

Let F be a minimal genus Seifert surface for $K = K(\alpha,\beta; p, q)$. By Lemma 30 the surface F can be isotoped in such a way that $F \cap \partial N(K_0)$ consists of $|lk(K_1, K_2)|$ preferred longitudes and $F \cap (S^3 - N(K_0))$ is made of $|lk(K_1, K_2)|$ components which are Seifert surfaces for K_0. Let $\tilde{F} = F \cap N(K_0)$, notice that once we determine the genus of $\tilde{F}$ the genus of F is obtained by adding $|lk(K_1, K_2)|$ times $(|p| - 1)(|q| - 1)/2$, which is the genus of the torus knot K_0.

The surface $F' = f^{-1}(\tilde{F})$ is an incompressible surface spanned by $L_{\beta/\alpha} = K_1 \cup K_2$ whose boundary consists of one component on $\partial n(K_2)$ and $|lk(K_1, K_2)|$ boundary components on $\partial n(K_1)$.

Lemma 34. *The boundary slope of surface F' on $\partial n(K_2)$ equals $-lk(K_1, K_2)^2 pq$ and the boundary slope of F' on $\partial n(K_1)$ equals $-1/pq$.*

Proof. Let l_1 and m_1 be the standard longitude and meridian of $\partial n(K_1)$ (chose any orientation of K_1) and let λ and μ the longitude and meridian of $\partial n(K_0)$, the morphism $f : \partial E(K_1) \to \partial n(K_0)$ sends m_1 to $pq\mu + \lambda$ (which is the fiber of the Seifert fibration of $E(K_0)$) and l_1 to μ so, the longitude λ is identified with $-pql_1 + m_1$ this means that the slope of F' on $\partial n(K_1)$ is equals to $\frac{-1}{pq}$.

Let $\partial_2 F'$ be the boundary of F' on $\partial n(K_2)$ and $\partial_1 F'$ be the one on $\partial n(K_1)$. It follows that $\partial_2 F'$ is homologous to $\partial_1 F'$ on $E(L_{\beta/\alpha})$. Observe that the inclusion $\partial N(K_2) \to E(L_{\beta/\alpha})$ induces an injection between the first homology groups, so $\partial_1 F'$ would be equivalent to only one class on $H_1(\partial N(K_2))$ that has to be $\partial_2 F'$.

Now, let l_2 and m_2 be the standard longitude and meridian of $\partial n(K_2)$ and $lk = lk(K_1, K_2)$. In $E(L_{\beta/\alpha})$, l_2 is homologous to $lk \cdot m_1$ (consider the disk bounded by l_2) and also l_1 is homologous to $lk \cdot m_2$. Then, $\partial_2 F' \sim \partial_1 F' \sim lk \cdot (-pql_1 + m_1) = -pq \cdot lk \cdot l_1 + lk \cdot m_1 = -pq \cdot lk^2 \cdot m_2 + l_2$, this implies that the boundary of F' in K_2 is homologous to $-pq \cdot lk^2 \cdot m_2 + l_2$ so its slope is $-pq \cdot lk^2$ □

In order to find the minimal genus of $K = K(\alpha, \beta; p, q)$, first we need to determine the minimal genus of the surface F' for the 2-bridge link $L_{\beta/\alpha} = K_1 \cup K_2$ with the above characteristics. That is to say, a surface F' with one boundary component on $\partial N(K_2)$ and $|lk(K_1, K_2)|$ boundary components on $\partial N(K_1)$, with boundary slopes as in Lemma 34, i.e, $\rho = 1$ and $\mu = |(pq)lk(K_1, K_2)|$. Since $p, q \neq 1$, then $\mu \neq 1$ even if $|lk(K_1, K_1)| = 1$. Observe that if $pq \geq 0$ then the boundary slopes turned out to be negative, and if $pq \leq 0$ they are positive. In both cases, the path associated to the continued fraction expansion $[r; 2r_1, ..., 2r_k]$ for β/α, with $r = 0$ or 1 and k odd, consists only of A and D−type edges by Lemma 11. By Proposition 21 it is possible to compute the genus of the orientable surface carried by such path. Moreover, when $r = 0$ the corresponding continued fraction is the one that gives rise to the surface with negative boundary slopes in both components of $E(L_{\beta/\alpha})$, by Corollary 23. When $r = 1$ we obtain a surface with positive boundary slopes on both components of $E(L_{\beta/\alpha})$, by Corollary 25. Summarizing we have the following result.

Theorem 35. *Let $L_{\beta/\alpha} = K_1 \cup K_2$ be the 2-bridge link given by the tunnel number one satellite knot $K(\alpha, \beta; p, q)$. And let F' be the essential surface for $L_{\beta/\alpha}$ that arises from a minimal genus Seifert surface F for $K(\alpha, \beta; p, q)$. Suppose $lk(K_1, K_2) \neq 0$. Then*

(1) *If $0 \leq \beta \leq \alpha$, $pq \geq 0$ and $[0; 2n_1, ..., 2n_j]$ is the unique continued fraction for β/α with j odd, the genus of F' is:*

$$\frac{1}{2}\left[\left(-1 + \sum_{k:odd} |n_k|\right)(|lk(K_1, K_2)pq| - 1) + (j + 1) - (|lk(K_1, K_2)| + 1)\right]$$

where $k \in \{1, ..., j\}$

(2) *If $0 \leq \beta \leq \alpha$, $pq \leq 0$ and $[1; 2m_1, ..., 2m_i]$ is the unique continued fraction for β/α with i odd, the genus of F' is:*

$$\frac{1}{2}\left[\left(-1 + \sum_{h:odd} |m_h|\right)(|lk(K_1, K_2)pq| - 1) + (i + 1) - (|lk(K_1, K_2)| + 1)\right]$$

where $h \in \{1, ..., i\}$

Corollary 36. *Let* $K = K(\alpha,\beta; p, q)$ *be a tunnel number one satellite knot, the genus of* K *is:*

$$g(K) = g(F') + |lk(K_1, K_2)|\frac{(|p| - 1)(|q| - 1)}{2}$$

Where F' *is as in* Theorem 35.

We can also determine if a satellite tunnel number one knot $K = K(\alpha,\beta; p, q)$ is fibered in the case that $lk(K_1, K_2) \neq 0$. Recall that the (p, q)-torus knot K_0 is fibered. A Seifert surface F for $E(K)$ is broken into pieces: $\tilde{F} = F \cap \partial n(K_0)$ and $|lk(K_1, K_2)|$ components which are Seifert surfaces for $E(K_0)$. These pieces are glued along a fiber of the Seifert fibration of the knot K_0. Thus, if $F' = f^{-1}(\tilde{F})$ is a fiber of a fibering of $E(L_{\beta/\alpha}) \to S^1$ then F will be a fiber of a fibering $E(K) \to S^1$. Theorem 28 part (1) gives us the condition to recognize when F' is a fiber for $E(L_{\beta/\alpha})$.

Proposition 37. *A tunnel number one satellite knot* $K(\alpha,\beta; p, q)$*, with* $lk(K_1, K_2) \neq 0$*, is fibered if and only if* β/α *has a continued fraction expansion of type* $[r; 2r_1, 2\epsilon_1, 2r_3, ..., 2\epsilon_k, 2r_k]$*, with* $r = 0$ *or* 1*,* $|\epsilon_i| = 1$ $(i = 1, ..., k)$ *and* k *odd.*

5.2. Torti-rational knots. Let $L_{\beta/\alpha} = K_1 \cup K_2$ be a 2-bridge link in S^3. Since K_1 is a trivial knot in S^3, K_2 can be considered as a knot in an unknotted solid torus V, the exterior of K_1. A copy of K_1 can be considered a meridian of V. Then by applying Dehn twists along a meridian disk of V in an arbitrary number of times, say r, we obtain a new knot K from K_2. We call this knot a *torti-rational knot* and it is denoted by $K(\beta/\alpha; r)$. In particular $K(\beta/\alpha; r)$ is contained in V. Let F be a minimal genus Seifert surface for $K(\beta/\alpha; r)$ of genus g. Consider the case when $lk(K_1, K_2) = 0$ if we can prove that $F \subset V$, this will let us compute the genus of F as in the case of satellite tunnel number one knots.

Lemma 38. *Let* F *be a minimal genus Seifert surface for the torti-rational knot* $K(\beta/\alpha; r)$*. Suppose* $lk(K_1, K_2) = 0$*, then* $F \subset V$*.*

Proof. Assume that $F \cap \partial V \neq \emptyset$, F can be isotoped to intersect ∂V in n longitudes and $F \cap (S^3 - V)$ consisting of n disjoint disks. Let $\tilde{F} = F \cap V$, after undoing the r Dehn twists along K_1, an essential spanning surface F' for $E(L_{\beta/\alpha})$ is obtained. The surface F' has one boundary component $\partial_2 F'$ parallel to K_2 and n boundary components $\partial_1 F'$ of slope $1/r$. Lemma 10 states that $|(1/r)\Sigma_v| = |lk(K_1, K_2)|$, then we have that $|\Sigma_v| = 0$. In particular the boundary components of F along $\partial n(K_1)$ have different orientations. Lemma 11 implies that if $\mu > 1$ and if a B-type saddle occurs then $|\Sigma_v| = 1$, which is a contradiction. Or if a D-type saddle appears then all boundary components of F' have the same orientation, which is not true. If $\mu = 1$ then $|\Sigma_v| = 1$, but it equals zero. Thus $\mu = 0$ implies that F' does not have boundary components on $\partial n(K_1)$, applying the r Dehn twist we recover F which is contained in V. □

Similarly to Lemma 32, the surface F' can be compressed meridionally g times to obtain a disk satisfying the conditions of Lemma 31. Thus we have the following result.

Proposition 39. *Let F be minimal Seifert genus surface for the torti-rational knot $K(\beta/\alpha; r)$ such that $lk(K_1, K_2) = 0$. The genus g of F is equal to one half the wrapping number of K_2 with respect to $E(K_1)$.*

Now consider the case $lk(K_1, K_2) \neq 0$, then $F \cap \partial V \neq \emptyset$. We will determine the genus of F in terms of the parameters β, α, r and $lk(K_1, K_2)$.

Theorem 40. *Let $K(\beta/\alpha; r)$ be a torti-rational knot and F a minimal genus Seifert surface for it. Suppose that $lk(K_1, K_2) \neq 0$. Then:*

(1) *If $r > 1$ and $[1; 2m_1, ..., 2m_i]$ is the unique continued fraction for β/α with i odd, the genus of F is:*

$$\frac{1}{2}\left[\left(-1 + \sum_{h:odd} |m_h|\right)(|lk(K_1, K_2)r| - 1) + (i+1) - (|lk(K_1, K_2)| + 1)\right]$$

where $h \in \{1, ..., i\}$

(2) *If $r < -1$ and $[0; 2n_1, ..., 2n_j]$ is the unique continued fraction for β/α with j odd, the genus of F is:*

$$\frac{1}{2}\left[\left(-1 + \sum_{k:odd} |n_k|\right)(|lk(K_1, K_2)r| - 1) + (j+1) - (|lk(K_1, K_2)| + 1)\right]$$

where $k \in \{1, ..., j\}$

(3) *If $|r| = 1$ and $|lk(K_1, K_2)| > 1$. Let $[s; 2r_1, ..., 2r_k]$ be the continued fraction expansion for β/α with $s = 0$ or 1 such that $k \geq 3$ and $|r_t| \geq 2$ for all t. The genus of F is:*

$$1 + \frac{(|lk(K_1, K_2)| + 1)(k-3)}{4}$$

(4) *If $|r| = 1$ and $|lk(K_1, K_2)| = 1$ and $[0; 2n_1, ..., 2n_j]$ and $[1; 2m_1, ..., 2m_i]$ are the continued fraction for β/α with j, i odd. The genus of F is:*

$$\min\left\{\frac{i-1}{4}, \frac{j-1}{4}\right\}$$

Proof. The surface F can be isotoped to intersect ∂V in n longitudes and $F \cap (S^3 - V)$ consisting of n disjoint disks. Let $\tilde{F} = F \cap V$, after undoing the r Dehn twists along K_1, an essential spanning surface F' for $E(L_{\beta/\alpha})$ is obtained. The surface F' has one boundary component $\partial_2 F'$ parallel to K_2 and n boundary components $\partial_1 F'$ of slope $1/r$. If we determine the genus of F' it will be the genus of F. By performing the corresponding r Dehn twists along K_1 we recover $\tilde{F}$, after capping of the n boundary components of $\tilde{F}$ we have F, thus F and F' have the same genus.

For the essential surface F', $\rho = 1$ and $\mu = |r|n$. By the formula of Lemma 10 we get $n = |lk(K_1, K_2)|$. The surface F' corresponds to some edge-path γ on a D_t diagram. Since $lk(K_1, K_2), \rho, n \neq 0$ then $t \neq 0, \infty$. If $\mu > 1$ Corollary 13 implies that γ is either an AD-edge-path or an AB-edge-path.

Suppose $r > 1$, the boundary components $\partial_1 F'$ have positive slope $1/r$, thus the slope is in correspondence with the slope given by the surface defined by the continued fraction expansion $[1; 2m_1, ..., 2m_i]$ for β/α, by Corollary 25. Applying Proposition 21 we obtain the result claimed in (1) .

Similarly, if $r < -1$ the boundary slopes of $\partial_1 F'$ are negative and by Corollary 23, F' is in correspondence with the path given by the continued fraction expansion $[0; 2n_1, ..., 2n_j]$. The genus of F' is given by Proposition 21 and hence we have proof (2).

If $|r| = 1$, then $\mu = n$. If $|lk(K_1, K_2)| > 1$ then γ is a minimal AB-edge-path. Let $[s; 2r_1, ..., 2r_k]$ be the continued fraction expansion for β/α with $s = 0$ or 1 and such that $k \geq 3$ and $r_t \geq 2$ for all t. The genus of F' is computed using Proposition 20. We have proved (3).

In the case that $|r| = 1$ and $|lk(K_1, K_2)| = 1$, the path γ is an A-edge-path. Let $[0; 2n_1, ..., 2n_j]$ and $[1; 2m_1, ..., 2m_i]$ be the continued fraction for β/α with j, i odd. Using Proposition 19, we can compute the genus of the two surfaces corresponding to both continued fractions. We pick the minimum between them, and we get part (4) of the Theorem. □

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